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ON THE GENERALIZED ORTHOGONALITY RELATION OF P.A.SCHIFF

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It has been assumed until now that Papkovich [1, 2] was the first author to derive the generalized orthogonality relation and to pose the problem of simultaneous expansion of two independent functions in series in homogeneous solutions. This problem has been dealt with within the framework of the plane problem of elasticity theory by Grinberg [3], Prokopov [4], Vorovich and Koval'chuk [5], and by several foreign authors whose studies are summarized in survey [6].

However, as was recently discovered, Papkovich's paper [1] gave impetus to studies of a problem whose history dates back to a variant of the three-dimensional problem of the theory of elasticity. We are referring to a paper by Schiff [8] (1883) which contains a derivation of the general orthogonality relation for the axisymmetric homogeneous solutions of the first fundamental problem of elasticity theory (that of an infinite hollow cylinder). Schiff used this relation to obtain the exact solution of the problem of axisymmetric straining of a finite solid cylinder with two arbitrary given functions at its end faces. One of these functions describes the shearing stresses; the other function is related to the volume expansion, but cannot be determined in advance from the mechanical conditions. In the same paper Schiff also solves for the first time the problem of torsion of a finite cylinder by forces distributed uniformly over its end faces. The only author who made any use of Schiff's method was Steklov [9]. The relegation of his method to oblivion is apparently attributable to its incorrect appraisal by Filon in his widely known paper [10].

We shall extend Schiff's relation to other forms of homogeneous boundary conditions and use the Fourier method to solve certain particular problems concerning a finite elastic cylinder. We shall also consider the possibility of exact satisfaction of either of the two given conditions at the end of a cylinder in the general case of loading when the other condition has been satisfied approximately.

1. Let us write out Schiff's solution, abbreviating it by the introduction of Papkovich-Neuber functions interrelated in a certain way. In the cylindrical coordinates r, z, φ the projections of the displacement vector in the axisymmetric case are given by the formulas [11]

$$u = B_1 - \frac{1}{4(1-\sigma)} \frac{\partial}{\partial r}(rB_1 + zB_2 + B_3)$$
$$w = B_2 - \frac{1}{4(1-\sigma)} \frac{\partial}{\partial z}(rB_1 + zB_2 + B_3) \tag{1.1}$$

where σ is the Poisson coefficient; the functions B_2 and B_3 satisfy the Laplace equation

$$\frac{\partial^2 B_n}{\partial r^2} + \frac{1}{r}\frac{\partial B_n}{\partial r} + \frac{\partial^2 B_n}{\partial z^2} = 0 \qquad (n = 2, 3, 4, 5) \tag{1.2}$$

and the function B_1 is the coefficient of $e^{i\varphi}$ in the expression for the harmonic function $\Delta(B_1 e^{i\varphi}) = 0$, i.e. it satisfies the equation

$$\frac{\partial^2 B_1}{\partial r^2} + \frac{1}{r}\frac{\partial B_1}{\partial r} - \frac{1}{r^2}B_1 + \frac{\partial B_1}{\partial z^2} = 0 \tag{1.3}$$

Let us set

$$B_1 = \frac{\partial B_4}{\partial r}, \qquad B_2 = 0, \qquad B_3 = 4(1-\sigma)(B_4 - B_5) \tag{1.4}$$

where B_4 and B_5 are harmonic functions. We shall limit ourselves to functions symmetric with respect to the middle plane $z = 0$ of the cylinder (the antisymmetric case can be analyzed in precisely similar fashion), setting

$$B_4 = \sum_{k=1}^{\infty} \rho_k(r) \operatorname{ch} m_k z, \qquad B_5 = \sum_{k=1}^{\infty} \eta_k(r) \operatorname{ch} m_k z \tag{1.5}$$

By virtue of (1.2), the functions $\rho_k(r)$ and $\eta_k(r)$ are the solutions of the Bessel equation

$$\rho_k''(r) + \frac{1}{r}\rho_k'(r) + m_k^2 \rho_k(r) = 0, \qquad \eta_k''(r) + \frac{1}{r}\eta_k'(r) + m_k^2 \eta_k(r) = 0 \tag{1.6}$$

The numbers m_k are determined by the boundary conditions. Substituting first (1.4) and then (1.5) into (1.1), we obtain

$$u = 2\sum_{k=1}^{\infty} \left[\eta_k'(r) + \frac{m_k^2 r \rho_k(r)}{4(1-\sigma)}\right] \operatorname{ch} m_k z \tag{1.7}$$

$$w = 2\sum_{k=1}^{\infty}\left[\eta_k(r) - \rho_k(r) - \frac{r\rho_k'(r)}{4(1-\sigma)}\right] m_k \operatorname{sh} m_k z \qquad \text{(cont.)}$$

Let us introduce the function $\varepsilon_k(r)$ (Omitting the argument for brevity),

$$\eta_k - \frac{r\rho_k'}{4(1-\sigma)} = \frac{\varepsilon_k + \rho_k}{2} \tag{1.8}$$

which satisfies the equation

$$\varepsilon_k'' + \frac{1}{r}\varepsilon_k' + m_k^2\varepsilon_k = \frac{m_k^2\rho_k}{1-\sigma} \tag{1.9}$$

The functions ε_k and ρ_k enable us to simplify expressions (1.7) for the displacements,

$$u = \sum_{k=1}^{\infty}(\varepsilon_k' + \rho_k')\operatorname{ch} m_k z, \qquad w = \sum_{k=1}^{\infty} m_k(\varepsilon_k - \rho_k)\operatorname{sh} m_k z \tag{1.10}$$

The components of the stress tensor become

$$\sigma_r = 2G\sum_{k=1}^{\infty}\left[\varepsilon_k'' + \rho_k'' - \frac{\sigma m_k^2\rho_k}{(1-\sigma)}\right]\operatorname{ch} m_k z \tag{1.11}$$

$$\sigma_\varphi = 2G\sum_{k=1}^{\infty}\left[\frac{\varepsilon_k' + \rho_k'}{r} - \frac{\sigma m_k^2\rho_k}{(1-\sigma)}\right]\operatorname{ch} m_k z \tag{1.12}$$

$$\sigma_z = 2G\sum_{k=1}^{\infty}\left[(\varepsilon_k - \rho_k)m_k^2 - \frac{\sigma m_k^2\rho_k}{(1-\sigma)}\right]\operatorname{ch} m_k z \tag{1.13}$$

$$\tau_{rz} = 2G\sum_{k=1}^{\infty}\varepsilon_k' m_k \operatorname{sh} m_k z \tag{1.14}$$

Let us show that the generalized orthogonality relation

$$\int_{r_1}^{r_2}(\varepsilon_j'\rho_k' + \varepsilon_k'\rho_j')\, r\, dr = 0, \quad j \neq k \tag{1.15}$$

obtained by Schiff in the absence of normal and shearing stresses at the boundary surfaces of a hollow infinite cylinder $r_1 \leqslant r \leqslant r_2$ is also valid under other homogeneous conditions.

We begin by considering the second fundamental problem of elasticity theory,

$$u = w = 0 \quad \text{for} \quad r = r_1 \ \text{and} \ r = r_2 \tag{1.16}$$

Let us introduce the function $\theta_k = \rho_k + \varepsilon_k$. By (1.6) and (1.9) this function satisfies equation

$$\theta_k'' + \frac{1}{r}\theta_k' + m_k^2\theta_k = \frac{m_k^2\rho_k}{1-\sigma} \tag{1.17}$$

By (1.10), boundary conditions (1.16) become

$$\theta_k' = 0,\ \theta_k - 2\rho_k = 0 \quad \text{for} \quad r = r_1 \ \text{and} \ r = r_2 \tag{1.18}$$

These conditions constitute a homogeneous system of four algebraic equations in the four arbitrary constants occurring in the functions η_k and ρ_k. The zeros of the determinant of the system determine the numbers m_k; its nontrivial solution gives three of the constants in terms of the fourth. Equation (1.17) yields

$$r\left[\frac{1}{r}(r\theta_k')'\right]' = r\left(\frac{m_k^2\rho_k'}{1-\sigma} - m_k^2\theta_k'\right) \tag{1.19}$$

Integration by parts yields

$$\int_{r_1}^{r_2} r\left[\frac{1}{r}(r\theta_j')'\right]'\theta_k'dr = [\theta_k'(r\theta_j')']_{r_1}^{r_2} - [\theta_j'(r\theta_k')']_{r_1}^{r_2} + \int_{r_1}^{r_2}\theta_j'\left[\frac{1}{r}(r\theta_k')'\right]' rdr$$

or, with allowance for the first condition of (1.18),

$$\int_{r_1}^{r_2}\left[\frac{1}{r}(r\theta_j')'\right]'\theta_k' rdr = \int_{r_1}^{r_2}\left[\frac{1}{r}(r\theta_k')'\right]'\theta_j' r\,dr \tag{1.20}$$

By virtue of this equation, expression (1.19) implies that

$$(m_j^2 - m_k^2)\int_{r_1}^{r_2}\theta_j'\theta_k' rdr = -\frac{1}{1-\sigma}\int_{r_1}^{r_2}(m_k^2\rho_k'\theta_j' - m_j^2\rho_j'\theta_k')\, r\,dr \tag{1.21}$$

On the other hand,

$$\int_{r_1}^{r_2}\theta_j'\theta_k' rdr = (\theta_j\theta_k' r)_{r_1}^{r_2} - \int_{r_1}^{r_2}\theta_j(r\theta_k')'\,dr = m_k^2\int_{r_1}^{r_2}\theta_j\left(r\theta_k - \frac{r\rho_k}{(1-\sigma)}\right)dr =$$

$$= m_k^2\int_{r_1}^{r_2}\theta_j\theta_k rdr + \frac{1}{1-\sigma}\int_{r_1}^{r_2}\theta_j(r\rho_k')'\,dr = m_k^2\int_{r_1}^{r_2}\theta_j\theta_k rdr +$$

$$+\frac{1}{1-\sigma}(r\theta_j\rho_k')_{r_1}^{r_2} - \frac{1}{1-\sigma}\int_{r_1}^{r_2} r\rho_k'\theta_j' dr \tag{1.22}$$

Interchanging the subscripts j and k, we obtain

$$\int_{r_1}^{r_2}\theta_j'\theta_k' rdr = m_j^2\int_{r_1}^{r_2}\theta_j\theta_k rdr + \frac{1}{1-\sigma}(r\theta_k\rho_j')_{r_1}^{r_2} - \frac{1}{1-\sigma}\int_{r_1}^{r_2}\rho_j'\theta_k' rdr \tag{1.23}$$

Let us multiply both sides of identities (1.22) and (1.23) by m_j^2 and m_k^2, respectively, and substract the second from the first,

$$(m_j^2 - m_k^2)\int_{r_1}^{r_2}\theta_j'\theta_k' rdr = \frac{1}{(1-\sigma)}(r\rho_k'\theta_j m_j^2 - r\rho_j'\theta_k m_k^2)_{r_1}^{r_2} -$$

$$-\frac{1}{1-\sigma}\int_{r_1}^{r_2}(m_j^2\rho_k'\theta_j' - m_k^2\rho_j'\theta_k')\,rdr \tag{1.24}$$

Equating the left sides of (1.21) and (1.24) and recalling the second condition of (1.18), we obtain

$$\frac{m_k^2 - m_j^2}{1-\sigma}\int_{r_1}^{r_2}(\varepsilon_j'\rho_k' + \varepsilon_k'\rho_j')\,rdr = \frac{2(m_j^2 - m_k^2)}{1-\sigma}\int_{r_1}^{r_2}\rho_k'\rho_j' rdr -$$

$$-\frac{2}{1-\sigma}(r\rho_k'\rho_j m_j^2 - r\rho_j'\rho_k m_k^2)_{r_1}^{r_2} = \frac{2}{1-\sigma}\int_{r_1}^{r_2}[(r\rho_j')'\rho_k m_k^2 - (r\rho_k')'\rho_j m_j^2]\,dr = 0$$

QED.

Let the hollow cylinder be subject to sliding support conditions, i.e. let $\tau_{rz} = u = 0$ for $r = r_1$ and $r = r_2$. We can use the same proof procedure in this case. Setting $\theta_k = \varepsilon_k$ and recalling that $\varepsilon_k' = \rho_k' = 0$ at the boundary of the cylinder for $r = r_1$ and $r = r_2$, we obtain

$$(m_j^2 - m_k^2)\int_{r_1}^{r_2}\varepsilon_j'\varepsilon_k' rdr = -\frac{1}{1-\sigma}\int_{r_1}^{r_2}(m_k^2\rho_k'\varepsilon_j' - m_j^2\rho_j'\varepsilon_k')\,rdr \tag{1.25}$$

$$\int_{r_1}^{r_2} \varepsilon_j'\varepsilon_k' r dr = m_j^2 \int_{r_1}^{r_2} \varepsilon_j\varepsilon_k r dr - \frac{1}{1-\sigma}\int_{r_1}^{r_2} \rho_j'\varepsilon_k' r dr \qquad (1.26)$$

$$(m_j^2 - m_k^2) \int_{r_1}^{r_2} \varepsilon_j'\varepsilon_k' r dr = -\frac{1}{1-\sigma}\int_{r_1}^{r_2} (m_j^2\rho_k'\varepsilon_j' - m_k^2\rho_j'\varepsilon_k') r\, dr \qquad (1.27)$$

instead of Eqs. (1.21), (1.23), (1.24), respectively.

Relations (1.25) and (1.27) immediately imply generalized orthogonality (1.15).

Finally, let the conditions at the boundary surfaces be $w = 0$, $\sigma_r = 0$. From (1.10) and (1.11) we have

$$\varepsilon_k = \rho_k, \quad \varepsilon_k' + \rho_k' + m_k^2 r\varepsilon_k = 0 \quad \text{for } r = r_1 \text{ and } r = r_2 \qquad (1.28)$$

Integrating by parts and recalling the first condition of (1.28), we obtain

$$\int_{r_1}^{r_2} r\left[\frac{1}{r}(r\varepsilon_j')'\right]'\varepsilon_k' dr = \int_{r_1}^{r_2} r\left[\frac{1}{r}(r\varepsilon_k')'\right]'\varepsilon_j' dr + \left[\frac{\sigma r}{1-\sigma}(\varepsilon_k' m_j^2\varepsilon_j - \varepsilon_j' m_k^2\varepsilon_k)\right]_{r_1}^{r_2}$$

This expression and the equation

$$r\left[\frac{1}{r}(r\varepsilon_k')'\right]' = r m_k^2\left(\frac{\rho_k'}{1-\sigma} - \varepsilon_k'\right)$$

imply that

$$(m_j^2 - m_k^2)\int_{r_1}^{r_2} \varepsilon_j'\varepsilon_k' r dr = -\frac{1}{1-\sigma}\int_{r_1}^{r_2} (m_k^2\rho_k'\varepsilon_j' - m_j^2\rho_j'\varepsilon_k') r dr -$$
$$-\left[\frac{\sigma r}{1-\sigma}(\varepsilon_k' m_j^2\varepsilon_j - \varepsilon_j' m_k^2\varepsilon_k)\right]_{r_1}^{r_2} \qquad (1.29)$$

Further, proceeding as in the derivation of (1.22) and (1.24), we arrive at the identities

$$\int_{r_1}^{r_2} \varepsilon_j'\varepsilon_k' r dr = m_k^2\int_{r_1}^{r_2} r\varepsilon_j\varepsilon_k dr + \frac{1}{1-\sigma}(r\varepsilon_j\rho_k')_{r_1}^{r_2} + (\varepsilon_j\varepsilon_k' r)_{r_1}^{r_2} - \frac{1}{1-\sigma}\int_{r_1}^{r_2} \rho_k'\varepsilon_j' r dr \qquad (1.30)$$

$$(m_j^2 - m_k^2)\int_{r_1}^{r_2} \varepsilon_j'\varepsilon_k' r dr = -\frac{1}{1-\sigma}\int_{r_1}^{r_2} [m_j^2\rho_k'\varepsilon_j' - m_k^2\rho_j'\varepsilon_k']\, r dr +$$
$$+\left[m_j^2 r\varepsilon_j\left(\frac{\rho_k'}{1-\sigma} + \varepsilon_k'\right) - m_k^2 r\varepsilon_k\left(\frac{\rho_j'}{1-\sigma} + \varepsilon_j'\right)\right]_{r_1}^{r_2} \qquad (1.31)$$

Equating the left sides of (1.29) and (1.31), we obtain

$$(m_j^2 - m_k^2)\int_{r_1}^{r_2} (\rho_k'\varepsilon_j' + \rho_j'\varepsilon_k') r dr - [m_j^2 r\varepsilon_j(\rho_k' + \varepsilon_k') - m_k^2\varepsilon_k r(\rho_j' + \varepsilon_j')]_{r_1}^{r_2} = 0$$

By virtue of the second condition of (1.28) the terms outside the integral in the sum amount to zero. This fact implies relation (1.15).

Without giving the proofs, we note that this relation is also valid for problems in which the four types of homogeneous boundary conditions at the inner and outer surfaces of the cylinder occur in various combinations. Specifically, those cases where the sliding support conditions are given at the inner surface also cover the problems concerning a solid cylinder. In fact, the displacements and stresses in a solid cylinder can be expressed in terms of Bessel functions of the first kind by virtue of their boundedness at the axis. The properties of Bessel functions of the first kind, (1.8) and (1.6) for $r = 0$ imply that $\varepsilon_k' = \rho_k' = 0$, or that $u = \tau_{rz} = 0$.

2. Let us consider the problem of equilibrium of a hollow elastic cylinder $r_1 \leqslant r \leqslant r_2$,

$-l \leqslant z \leqslant l$ at whose end faces we have the following given tangential displacements and normal stresses symmetric with respect to the middle plane $z = 0$:

$$u = f_1(r), \quad \sigma_z = f_2(r) \quad \text{for} \quad z = \pm l, \; r_1 \leqslant r \leqslant r_2 \tag{2.1}$$

The boundary conditions at cylindrical surfaces can always be reduced to homogeneous conditions by suitable alteration of the functions $f_1(r)$ and $f_2(r)$. The homogeneous conditions, whose form we shall for the moment leave unspecified, determine the characteristic numbers m_k and relate the functions $\varepsilon_k(r)$ and $\rho_k(r)$, leaving only one of the four arbitrary constants a_k. Let $\varepsilon_k(r) = a_k\gamma_k(r), \quad \rho_k = a_k\delta_k(r)$

so that the functions $\gamma_k(r)$ and $\delta_k(r)$ satisfy the same relations (1.6), (1.9), (1.15).

By virtue of conditions (2.1), formulas (1.13), (1.10) and Eq. (1.9) yield

$$\sigma_z = -2G\sum_{k=1}^{\infty} a_k \frac{1}{r}(r\gamma_k')' \operatorname{ch} m_k l = f_2(r), \quad u = \sum_{k=1}^{\infty} a_k(\gamma_k' + \delta_k') \operatorname{ch} m_k l = f_1(r)$$

Rewriting these equations in the form

$$F_1'(r) = (rf_1(r))' - \frac{rf_2(r)}{2G} = \sum_{k=1}^{\infty} a_k (r\delta_k')' \operatorname{ch} m_k l \tag{2.2}$$

$$F_2'(r) = -\frac{rf_2(r)}{2G} = \sum_{k=1}^{\infty} a_k (r\gamma_k')' \operatorname{ch} m_k l \tag{2.3}$$

and making use of generalized orthogonality relation (1.15), we obtain the coefficients a_k for any homogeneous conditions,

$$a_k = [2\operatorname{ch} m_k l \int_{r_1}^{r_2} \gamma_k'\delta_k' r\,dr]^{-1} \int_{r_1}^{r_2} (F_1\gamma_k' + F_2\delta_k')\,dr$$

Here the functions F_1 and F_2 are the primitives of the functions F_1' and F_2'. This generally makes it difficult to determine the coefficients a_k. If rigid fastening ($\varepsilon_k' = -\rho_k'$, $\varepsilon_k = \rho_k$) or sliding support ($\varepsilon_k' = \rho_k' = 0$) conditions (or their combinations) are given at the boundary surfaces $r = r_1$ and $r = r_2$ of the cylinder, then the coefficients a_k can be found from another formula which does not contain primitives. Let us multiply Eq. (2.2) by γ_j, add to it Eq. (2.3) multiplied by δ_j, and integrate the result by parts. By virtue of (1.15) we obtain

$$\int_{r_1}^{r_2} (F_1'\gamma_j + F_2'\delta_j)\,dr = \sum_{k=1}^{\infty} [a_k(\gamma_j\delta_k' + \gamma_k'\delta_j) r \operatorname{ch} m_k l]_{r_1}^{r_2} + 2a_j \operatorname{ch} m_j l \int_{r_1}^{r_2} \gamma_j'\delta_j' r\,dr$$

The above conditions make the terms under the summation sign amount to zero. Hence,

$$a_k = [2\operatorname{ch} m_k l \int_{r_1}^{r_2} \gamma_k'\delta_k' r\,dr]^{-1} \int_{r_1}^{r_2} (F_1'\gamma_k + F_2'\delta_k)\,dr$$

The same procedure can be used to solve the problem of a cylinder with the shearing stresses and normal displacements given at its ends, i.e.

$$\tau_{rz} = f_1(r), \quad w = f_2(r) \quad \text{for} \quad z = \pm l, \; r_1 \leqslant r \leqslant r_2$$

Formulas (1.14) and (1.10) imply that

$$f_1(r) = 2G\sum_{k=1}^{\infty} a_k\gamma_k' m_k \operatorname{sh} m_k l, \quad f_2(r) = \sum_{k=1}^{\infty} a_k(\gamma_k' - \delta_k') m_k \operatorname{sh} m_k l$$

Generalized orthogonality relation (1.15) yields

$$a_k = [4Gm_k \operatorname{sh} m_k l \int_{r_1}^{r_2} \gamma_k'\delta_k' r dr]^{-1} \int_{r_1}^{r_2} [(f_1 - 2Gf_2')\gamma_k' + f_1\delta_k'] r dr$$

This formula contains the derivative of the function $f_2(r)$ only, so that the solution does not take account of the constant component of the normal displacement and is valid for problems which have the homogeneous condition $w = 0$ given at one or both of the cylindrical surfaces. To render it suitable in the case where the normal and shearing stresses are given at the side surface we need merely add the elementary solution of the problem of a cylinder under normal tensile stresses σ distributed uniformly over its end faces.

3. Following Papkovich [1], we can also apply the generalized orthogonality relation to the fundamental problems of elasticity theory concerning a finite hollow cylinder. This yields a solution closest to the exact one in the sense that of the eight conditions given at the four boundary surfaces six are satisfiable by the Fourier method, while the remaining two (one at each end) are satisfied approximately. Other familiar methods can be used to satisfy four or all eight of the conditions.

For example, let us consider the first fundamental problem of elasticity theory for a finite solid cylinder $0 \leqslant r \leqslant r_1$, $-l \leqslant z \leqslant l$ with homogeneous conditions at its side surface and symmetric conditions at its ends.

$$\sigma_r = \tau_{rz} = 0 \quad \text{for} \quad r = r_1 \tag{3.1}$$

$$\sigma_z = f_1(r), \quad \tau_{rz} = f_2(r) \quad \text{for} \quad z = \pm l \quad (0 \leqslant r \leqslant r_1) \tag{3.2}$$

We assume that the principal stress vector at the ends of the cylinder has been reduced to zero by addition of the suitable elementary solution. Formulas (1.11), (1.14) and Eqs. (1.8), (3.1) yield the familiar characteristic equation which determines the numbers m_k

$$[2(1-\sigma) - m_k^2 r_1^2] J_1^2(m_k r_1) - m_k^2 r_1^2 J_0^2(m_k r_1) = 0$$

and the relationship between the functions ε_k and ρ_k,

$$\varepsilon_k = a_k\gamma_k, \quad \rho_k = a_k\delta_k, \quad \delta_k = J_0(m_k r) \tag{3.3}$$

$$\gamma_k = \frac{[(3-2\sigma) J_1(m_k r_1) - r_1 m_k J_0(m_k r_1)] J_0(m_k r)}{2(1-\sigma) J_1(m_k r_1)} + \frac{r m_k J_1(m_k r)}{2(1-\sigma)} - J_0(m_k r)$$

By (3.2), (1.13) and (1.9) we have

$$f_1(r) = 2G \sum_{k=1}^{\infty} a_k m_k^2 \left(\gamma_k - \frac{\delta_k}{1-\sigma}\right) \operatorname{ch} m_k l \tag{3.4}$$

Let us introduce the function

$$H(r) = 2G \sum_{k=1}^{\infty} a_k m_k^2 \gamma_k' \operatorname{ch} m_k l \tag{3.5}$$

which remains unknown for the time being, and subtract from it the function $f_1'(r)$. Recalling (3.4), we obtain

$$H(r) - f_1'(r) = \frac{2G}{1-\sigma} \sum_{k=1}^{\infty} a_k m_k^2 \delta_k' \operatorname{ch} m_k l \tag{3.6}$$

Now let us multiply (3.5) by $r\delta_j'$, add it to Eq. (3.6) multiplied by $(1-\sigma) r\gamma_j'$, and integrate the result over r from 0 to r_1.

By virtue of (1.5) we have

$$a_k = [4Gm_k^2 \operatorname{ch} m_k l \int_0^{r_1} \gamma_k'\delta_k' r dr]^{-1} \int_0^{r_1} [(1-\sigma)\gamma_k' (H(r) - f_1'(r)) + H(r)\delta_k'] r dr \tag{3.7}$$

Let us write out the function $H(r)$ as an expansion in some complete system of functions $\psi_n(r)$,

$$H(r) = \sum_{n=1}^{\infty} c_n \psi_n(r) \tag{3.8}$$

We can find the coefficients c_n by means of the second boundary condition of (3.2). Substituting (3.8) into (3.7), then (3.7) into (1.14), and changing the order of summation, we arrive at the relation

$$\sum_{n=1}^{\infty} c_n S_n(r) = T(r) \tag{3.9}$$

$$S_n(r) = \sum_{k=1}^{\infty} \frac{\operatorname{th} m_k l}{2m_k} \left[\int_0^{r_1} (\delta_k' + (1-\sigma)\gamma_k)\psi_n r\,dr \right] \left[\int_0^{r_1} \gamma_k' \delta_k' r\,dr \right]^{-1} \gamma_k' \tag{3.10}$$

$$T(r) = f_2(r) + \frac{1-\sigma}{2} \sum_{k=1}^{\infty} \frac{\operatorname{th} m_k l}{m_k} \left[\int_0^{r_1} f_1'(r)\gamma_k' r\,dr \right] \left[\int_0^{r_1} \gamma_k' \delta_k' r\,dr \right]^{-1} \gamma_k' \tag{3.11}$$

from which we can determine the required coefficients c_n by the orthogonalization process of Schmidt.

It is convenient to take the eigenfunctions of the Sturm-Liouville problem for the Bessel equations as our $\psi_n(r)$. This allows us to compute the corresponding integrals in formula (3.10) with the aid of tables of indefinite integrals. The suitable boundary conditions for the Sturm-Liouville problem can be chosen readily on the basis of homogeneous conditions (3.1) and expansion (3.5) of the function $H(r)$. For example, in our problem the condition $\varepsilon_k' = 0$ or, according to (3.5), $H(r) = 0$ for $r = 0$ and $r = r_1$ means that that the eigenfunctions can be conveniently taken in the form

$$\psi_n''(r) + \frac{1}{r}\psi_n'(r) + \left(p_n^2 + \frac{1}{r^2}\right)\psi_n(r) = 0$$

$$\psi_n(r) = 0 \quad \text{for } r = 0 \text{ and } r = r_1$$

i.e. $\psi_n(r) = J_1(p_n r)$, where the numbers p_n satisfy condition $J_1(p_n r_1) = 0$.

In conclusion we note that any other condition at the ends of the cylinder can be exactly satisfied by the same procedure. For example, if this condition is for the shearing stress τ_{rz}, then instead of (3.5) we set

$$H(r) = 2G \sum_{k=1}^{\infty} a_k m_k \delta_k' \operatorname{sh} m_k l$$

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NOTE ON THE ERRORS APPEARING IN THE BOOK "CONFLUENT HYPERGEOMETRIC FUNCTIONS" BY L. J. SLATER

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Erroneous formulas of expansion of the Kummer and Whittaker functions of the first kind in terms of cylindrical functions given in a monograph [1] by Slater, are corrected.

Expansions of functions ${}_1F_1(a, b, x)$ and $M_{k,m}(x)$ into series in cylindrical functions are found useful when tables covering the required interval of variation of parameters are not available. Slater gives such expansions in [1], unfortunately all four formulas appearing in their final form in Sect. 2.7.3 are erroneous.

Fallacy of Formulas (2.7.14) and (2.7.16) becomes obvious on applying them to already known cases. Indeed, when $a = n + 1/2$ and $b = 2n + 1$, Formula (2.7.14) yields

$${}_1F_1[n + 1/2, 2n + 1, x] = 2^{2n}\Gamma(n)e^{1/2 x}x^{-n}I_n(1/2 x)$$

which contradicts the particular Formula (2.7.1). On putting $k = 0$, Formula (2.7.16) yields

$$M_{0,m}(x) = 2^{2m}\Gamma(m)x^{1/2}I_m(1/2\, x)$$

which in turn contradicts the exact Formula (1.8.11) (see also (9.235) of [2]).

On checking we have found that the error was caused by the incorrect computation in [1] of the function ${}_0F_1(;b, x)$ on passing from Formula (2.7.10) to (2.7.14) and (2.7.15), and the function was transferred, as it stood, into (2.7.16) and (2.7.17). Since the same error appears in all four formulas, we shall compute ${}_1F_1(a, b, x)$ and give the exact result for $M_{k,m}(x)$.

By (2.7.11) we find

$${}_0F_1[; b - a + 1/2 + n, (1/4 x)^2] = \Gamma(b - a + 1/2 + n)(1/4 x)^{a-b+1/2-n}I_{b-a-1/2+n}(1/2 x)$$